

# The Minimal Non-Koszul $A(\Gamma)$

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## Abstract

The algebras  $A(\Gamma)$ , where  $\Gamma$  is a directed layered graph, were first constructed by I. Gelfand, S. Sercone, V. Retakh and R. Wilson. These algebras are generalizations of the algebras  $Q_n$ , which are related to factorizations of non-commutative polynomials. It was conjectured that these algebras were Koszul. In 2008, T. Cassidy and B. Shelton found a counterexample to this claim, a non-Koszul  $A(\Gamma)$  corresponding to a graph  $\Gamma$  with 18 edges and 11 vertices.

We produce an example of a directed layered graph  $\Gamma$  with 13 edges and 9 vertices which produces a non-Koszul  $A(\Gamma)$ . We also show this is the minimal example with this property.

## 1 Introduction

The relationship between the factorizations and coefficients of a non-commutative polynomial is described in terms of pseudo-roots by the non-commutative version of Vieta's theorem [4]. The algebra  $Q_n$  of the pseudo-roots for a polynomial of degree  $n$  has been described and studied by Gelfand, Retakh and Wilson in [5]. These algebras are quadratic and Koszul, and their corresponding dual algebras  $Q_n^!$  have finite dimension.

The generators of this algebra correspond to elements of  $B_n$ , the boolean lattice of all subsets of an  $n$ -element set. This construction was then generalized to form the class of algebras  $A(\Gamma)$ , each of which is determined by a layered graph  $\Gamma$ . The algebras  $Q_n$  are simply the algebras  $A(\Gamma)$ , where  $\Gamma$  equals  $B_n$ .

Depending on a condition called uniformity of the graph  $\Gamma$ , the algebra  $A(\Gamma)$  may be quadratic [8], thus leading to the question of which of these algebras are Koszul. It was discovered that the algebras were Koszul for the boolean lattice  $B_n$ , simplicial complexes and complete layered graphs with arbitrary numbers of vertices at each level [9, 10]. It was also conjectured that all algebras  $A(\Gamma)$  were Koszul. This was shown not to be the case when Cassidy and Shelton found the first example of a non-Koszul  $A(\Gamma)$  [3]; see Figure 1.

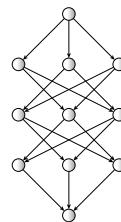


Figure 1: A Non-Koszul  $A(\Gamma)$  with Eleven Vertices

This led to the question of what the smallest  $\Gamma$  with a quadratic non-Koszul  $A(\Gamma)$  might be. In 2010, Retakh, Sercone, and Wilson found that when one edge is removed from a particular layer, we retain both uniformity and non-Koszulity of  $A(\Gamma)$  [7] (see Figure 2).

By computer, we found that we could extend the example further, removing one vertex from the second highest layer though this was not minimal; see Figure 3. In this paper we now produce the smallest non-

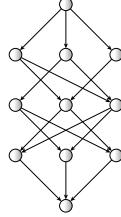


Figure 2: Another Non-Koszul  $A(\Gamma)$  with Eleven Vertices

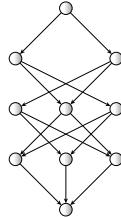


Figure 3: A Non-Koszul  $A(\Gamma)$  with Ten Vertices

Koszul example. We named this graph  $H$  after HiLGA,<sup>1</sup> the program we wrote to study algebras of the form  $A(\Gamma)$ . Computers are not needed for showing either the minimality in terms of number of vertices, or the non-Koszulity of  $H$ .

## 2 Theory

Let  $\Gamma = (V, E)$  be a directed graph, where  $V = \coprod_{i=0}^N V_i$  and  $V_i \neq \emptyset$  for  $i \in \{0, 1, \dots, N\}$ . We call these  $V_i$  the *layers* of  $\Gamma$ . For each vertex  $v$ , we write  $|v| = i$ , and say that  $v$  has height  $i$  if  $v \in V_i$ . For each edge, let  $t(e)$  denote the tail of  $e$  and  $h(e)$  denote the head of  $e$ . We say that  $\Gamma$  is a *layered graph of height  $N$*  if  $|t(e)| = |h(e)| + 1$  for all  $e \in E$ .

Given a layered graph, we construct an algebra  $A(\Gamma)$  as follows: Begin with the free algebra  $T(E)$  generated by the edges of  $\Gamma$ . Let  $\pi$  and  $\pi'$  be any two paths  $\pi = (e_1, e_2, \dots, e_k)$  and  $\pi' = (f_1, f_2, \dots, f_k)$  with the same starting and finishing vertices. For every such pair  $\pi, \pi'$ , we impose the relation

$$(t - e_1)(t - e_2) \cdots (t - e_k) = (t - f_1)(t - f_2) \cdots (t - f_k)$$

where  $t$  is a formal parameter commuting with all edges in  $E$ . Matching the coefficients of  $t$  gives us a collection of relations. Let  $I$  be the collection of all of these relations for any two paths meeting our conditions. Then  $A(\Gamma)$  is  $T(E)/I$ .

The standard definition of Koszulity requires a quadratic algebra. Not all algebras of the form  $A(\Gamma)$  are quadratic, but there is a condition on  $\Gamma$  which guarantees  $A(\Gamma)$  is quadratic. Two vertices  $v, v'$  of the same layer  $l$  are *connected by an up-down sequence* if there is a sequence  $v = v_0, v_1, \dots, v_k = v'$  of vertices of level  $l$  with the following property: For each  $i$  in  $\{1, \dots, k\}$  there are edges  $e$  and  $f$  so  $t(e) = v_{i-1}$ ,  $t(f) = v_i$  and  $h(e) = h(f)$ . A layered graph  $\Gamma$  is said to be *uniform* if for any pair of edges  $e, e'$  with a common tail, their heads are connected by an up down sequence  $v_0, v_1, \dots, v_k$  with  $v_i$  adjacent to  $t(e)$  for  $i \in \{1, \dots, k\}$ .

It has been shown in [8] that if  $\Gamma$  is a uniform layered graph then  $A(\Gamma)$  is a quadratic algebra. If  $\Gamma$  has a unique minimal vertex of layer zero then there is a construction giving us a new algebra we call  $B(\Gamma)$ . The algebras  $B(\Gamma)$  can be presented by generators  $u \in V^+ = \coprod_{i=1}^N V_i$  and relations

1.  $u \cdot w = 0$  if there is no edge from  $u$  to  $w$
2.  $u \cdot \sum_{w \in S(u)} w = 0$  where  $S(u)$  is the collection of vertices  $w$  for which there is an edge from  $u$  to  $w$ .

<sup>1</sup>The name HiLGA comes from “Hilbert Series of Layered Graph Algebras”.

The algebras  $B(\Gamma)$  are the dual algebras of the associated graded algebras of  $A(\Gamma)$  under a certain filtration. They have the same Hilbert series as the actual dual of  $A(\Gamma)$  and when it exists,  $B(\Gamma)$  is Koszul if and only if  $A(\Gamma)$  is [7].

### 3 A Layered Graph $\Gamma$ on Nine Vertices with non-Koszul $A(\Gamma)$

We introduce a layered graph  $H$  on nine vertices; see Figure 4.

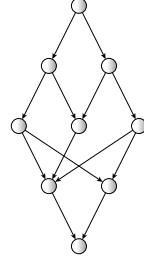


Figure 4: The Poset  $H$

**Theorem 3.1.** *The algebra  $A(\Gamma)$  is not Koszul for  $\Gamma = H$ .*

*Proof.* We use the numerical Koszulity test from theorem 4.2.1 in [7]. Considering the  $i = 4$  case we know that  $A(\Gamma)$  is not numerically Koszul if  $\dim(H^{-1}(\Gamma_{a,4})) - \dim(H^0(\Gamma_{a,4})) + \dim(H^1(\Gamma_{a,4})) \neq 0$  where the layer of  $a$  is greater than or equal to 4. Thus  $a$  must be the maximal vertex in  $H$  so  $\Gamma_{a,4}$  is the graph in Figure 5.

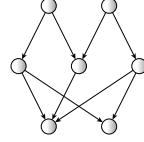


Figure 5: The Graph  $\Gamma_{a,4}$

By the Euler-Pointcare formula, we know  $\sum(-1)^i \dim(H^i) = \sum(-1)^j \dim(C^j)$ . Thus we have

$$\begin{aligned} \dim(H^{-1}(\Gamma_{a,4})) - \dim(H^0(\Gamma_{a,4})) + \dim(H^1(\Gamma_{a,4})) &= \\ \dim(C^{-1}(\Gamma_{a,4})) - \dim(C^0(\Gamma_{a,4})) + \dim(C^1(\Gamma_{a,4})) - \dim(C^2(\Gamma_{a,4})) + \dim(H^2(\Gamma_{a,4})) &\geq \\ \dim(C^{-1}(\Gamma_{a,4})) - \dim(C^0(\Gamma_{a,4})) + \dim(C^1(\Gamma_{a,4})) - \dim(C^2(\Gamma_{a,4})) & \\ = 1 - 7 + 13 - 6 &= 1. \end{aligned}$$

This shows that  $H$  is not numerically Koszul. Since Koszulity implies numerical Koszulity,  $H$  is not Koszul as well.  $\square$

### 4 The Minimal non-Koszul $A(\Gamma)$

We now wish to show that  $H$  is the unique minimal example. We must restrict ourselves to uniform  $\Gamma$  in order to guarantee that the algebra is quadratic<sup>2</sup>.

<sup>2</sup>There is still room, however, to generalize here using one of the different definitions of Koszulity for non-quadratic algebras.

We will need part of a proposition of Polishchuk and Positselksi found in [6] which we summarize here for clarity.

**Proposition 4.1.** *Let  $W$  be a vector space and  $X_1, \dots, X_N \subset W$  be a collection of its subspaces. Then the following are equivalent:*

1. *The collection  $X_1, \dots, X_N$  is distributive.*
2. *There exists a direct sum decomposition  $W = \bigoplus_{\eta \in H} W_\eta$  of the vector space  $W$  such that each of the subspaces  $X_i$  is the sum of a set of subspaces  $W_\eta$ .*

**Lemma 4.1.** *Suppose the vector subspaces  $W_1, W_2, \dots, W_M$  and  $X_1, X_2, \dots, X_N$  generate distributive lattices in  $Y$  and  $Z$  respectively. Then the subspaces  $W_1 \otimes Z, W_2 \otimes Z, \dots, W_M \otimes Z, Y \otimes X_1, Y \otimes X_2, \dots, Y \otimes X_N$  generate a distributive lattice in  $Y \otimes Z$ .*

*Proof.* By proposition 4.1, there exist index sets  $\alpha, \beta$  and decompositions  $Y = \bigoplus_{i \in \alpha} T_i$  and  $Z = \bigoplus_{i \in \beta} U_i$  so that for every  $i \in \{1, \dots, M\}$  and  $j \in \{1, \dots, N\}$  there are sets  $\alpha_i \subseteq \alpha$  and  $\beta_j \subseteq \beta$  so that  $W_i = \bigoplus_{k \in \alpha_i} T_k$  and  $X_j = \bigoplus_{k \in \beta_j} U_k$ . We can construct the direct sum decomposition  $Y \otimes Z = \bigoplus_{i \in \alpha, j \in \beta} (T_i \otimes U_j)$ . Then for any  $i \in \{1, \dots, M\}$ ,  $W_i \otimes Z = \bigoplus_{j \in \beta} (T_i \otimes U_j)$  and  $Y \otimes X_i = \bigoplus_{j \in \beta} (T_i \otimes U_j)$ . Referring to proposition 4.1 once more completes the proof.  $\square$

In order to show a quadratic algebra is Koszul, we need to show that for any positive integer  $n$ , the collection of subspaces  $V^i RV^{n-i-2}$  generates a distributive lattice in  $V^n$ . We repeat the following lemma from Serconek and Wilson's paper [10] which will allow us to reduce the problem to one where we check for distributivity inside a much smaller vector space.

**Lemma 4.2.** *Let  $V = \sum_{i \in I} V_{[i]}$  be a graded vector space and  $\{X_j | j \in J\}$  be a collection of subspaces of  $V$ . Assume that each  $X_j$  is graded,  $X_j = \sum_{i \in I} X_{j,[i]}$ , and  $X_{j,[i]} = X_j \cap V_{[i]}$ . Then  $\{X_j | j \in J\}$  generates a distributive lattice in  $V$  if and only if, for every  $i \in I$ ,  $\{X_{j,[i]} | j \in J\}$  generates a distributive lattice in  $V_{[i]}$ .*

**Lemma 4.3.** *Let  $A$  be a quadratic algebra where the generators are partitioned into the disjoint spaces  $V_1, V_2, \dots, V_N$ . Suppose every relation is contained inside the space  $V_{i+1}V_i$  for some  $i \in \{1, \dots, n-1\}$ . Consider the  $\mathbb{Z}^n$  grading of  $V^n$  where  $x_1x_2 \dots x_n$  is in  $V_{[(z_1, z_2, \dots, z_n)]}^n$  if and only if  $x_i \in V_i$  for all  $i \in \{1, \dots, n\}$ .*

*The collection  $\{RV^{n-2}, \dots, V^{n-2}R\}$  generates a distributive lattice in  $V^n$  if and only if  $\{RV_{[(n, n-1, \dots, 1)]}^{n-2}, VRV_{[(n, n-1, \dots, 1)]}^{n-3}, \dots, V^{n-2}R_{[(n, n-1, \dots, 1)]}\}$  generates a distributive lattice in  $V_{[(n, n-1, \dots, 1)]}^n$ .*

*Proof.* By lemma 4.2 we know that  $\{RV^{n-2}, \dots, V^{n-2}R\}$  generates a distributive lattice in  $V^n$  if and only if  $\{RV_{[(z_n, z_{n-1}, \dots, z_1)]}^{n-2}, VRV_{[(z_n, z_{n-1}, \dots, z_1)]}^{n-3}, \dots, V^{n-2}R_{[(z_n, z_{n-1}, \dots, z_1)]}\}$  generates a distributive lattice in  $V_{[(z_n, z_{n-1}, \dots, z_1)]}^n$  for every  $(z_n, z_{n-1}, \dots, z_1)$ . Thus we only need to show that if  $\{RV_{[(n, n-1, \dots, 1)]}^{n-2}, VRV_{[(n, n-1, \dots, 1)]}^{n-3}, \dots, V^{n-2}R_{[(n, n-1, \dots, 1)]}\}$  generates a distributive lattice in  $V_{[(n, n-1, \dots, 1)]}^n$  then  $\{RV_{[(z_n, z_{n-1}, \dots, z_1)]}^{n-2}, VRV_{[(z_n, z_{n-1}, \dots, z_1)]}^{n-3}, \dots, V^{n-2}R_{[(z_n, z_{n-1}, \dots, z_1)]}\}$  generates a distributive lattice in  $V_{[(z_n, z_{n-1}, \dots, z_1)]}^n$  for every  $(z_n, z_{n-1}, \dots, z_1)$ .

Consider the collection  $\{V_{z_n} V_{z_{n-1}} V_{z_{n-2}} \dots V_{z_{b+1}} W_i V_{z_{a-1}} V_{z_{a-2}} \dots V_{z_1}\}_{i \in \{a+1, \dots, b\}}$  where  $W_i = V_b V_{b-1} \dots V_{i+1} R_i V_i \dots V_{a+1} V_a$ . It is enough to show that this collection is distributive in  $V_{[(z_n, z_{n-1}, \dots, z_2, z_1)]}^n$  where  $z_{i+1} = z_i + 1$  for  $i \in \{a, \dots, b-1\}$  because then we can string these increasing runs together using lemma 4.1 to complete the proof.  $\square$

By our assumption, we do know that the collection  $\{V_n V_{n-1} V_{n-2} \dots V_{b+1} W_i V_{a-1} V_{a-2} \dots V_1\}_{i \in \{a+1, \dots, b\}}$  is distributive in  $V_{[(n, n-1, \dots, 2, 1)]}^n$  which implies the collection  $\{W_i\}_{i \in \{a+1, \dots, b\}}$  is distributive in  $V_{[(b, b-1, \dots, a+1, a)]}^{b-a+1}$ . This implies the distributivity of  $\{V_{z_n} V_{z_{n-1}} V_{z_{n-2}} \dots V_{z_{b+1}} W_i V_{z_{a-1}} V_{z_{a-2}} \dots V_{z_1}\}_{i \in \{a+1, \dots, b\}}$  in  $V_{[(z_n, z_{n-1}, \dots, z_2, z_1)]}^n$  completing the proof.  $\square$

**Theorem 4.1.** *Suppose  $\Gamma$  is a uniform layered graph on  $V = \bigcup_{i=0}^N V_i$  with exactly one vertex at layer  $k$ . Let  $\Gamma_0$  be the induced subgraph of  $\Gamma$  on  $\bigcup_{i=0}^k V_i$  and  $\Gamma_1$  be the induced subgraph of  $\Gamma$  on  $\bigcup_{i=k}^N V_i$ . Then  $A(\Gamma)$  is Koszul if  $A(\Gamma_0)$  and  $A(\Gamma_1)$  are Koszul.*

*Proof.* We work in terms of the associated graded algebra of  $A(\Gamma)$  (equal to  $B(\Gamma)^\dagger$ ) so that the relations meet the hypotheses of lemma 4.3. The lemma allows us to assume the height decreases at each step.

As  $B(\Gamma_1)^\dagger$  is Koszul we know that

$$\{(\otimes_{i=j+1}^N V_i) \otimes R_j \otimes (\otimes_{i=k+1}^{j-2} V_i)\}_{j \in \{k+2, \dots, N\}}$$

is distributive in  $Y = \otimes_{i=k+1}^N V_i$ . As  $B(\Gamma_0)^\dagger$  is Koszul we know that

$$\{(\otimes_{i=j+1}^k V_i) \otimes R_j \otimes (\otimes_{i=1}^{j-2} V_i)\}_{j \in \{2, \dots, k\}}$$

is distributive in  $Z = \otimes_{i=1}^k V_i$ .

Applying lemma 4.1 gives us the distributivity of the collection

$$\{(\otimes_{i=j+1}^N V_i) \otimes R_j \otimes (\otimes_{i=1}^{j-2} V_i)\}_{j \in \{2, \dots, k\} \cup \{k+2, \dots, N\}}$$

in  $\otimes_{i=1}^N V_i$ . Since  $R_{k+1} = 0$  this implies distributivity for  $j \in \{2, \dots, N\}$  thus showing  $B(\Gamma)^\dagger$  and thus  $A(\Gamma)$  is Koszul.

□

We can now use this to reduce the number of cases we need to consider. Say that a layered graph  $\Gamma$  is a  $[z_0, z_1, \dots, z_N]$ -graph if  $|V_i| = z_i$  for each  $i$ , where  $V_i$  is the collection of vertices of layer  $i$ .

**Proposition 4.2.** *Any uniform layered graph with unique minimal and maximal elements and non-Koszul  $A(\Gamma)$  has at least nine vertices.*

*Proof.* Suppose the  $[1, z_1, z_2, \dots, z_{n-1}, 1]$ -graph is minimal in number of vertices amongst graphs with non-Koszul  $A(\Gamma)$ . Then  $z_i > 1$  for all  $0 < i < N$ . Otherwise we could use theorem 4.1 to find a smaller example. We also know from [7] that no example of a non-Koszul  $A(\Gamma)$  exists for graphs of height three or less. This leaves one possibility for a non-Koszul  $A(\Gamma)$  with under nine vertices: that of a  $[1, 2, 2, 2, 1]$ -graph.

There are only ten such graphs with unique maximal and minimal vertices [2], and of those only five [1] are uniform; see Figure 6. It is easy to check these five cases, by either hand or computer to see these are

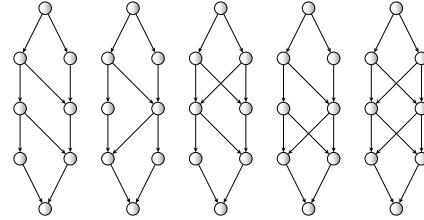


Figure 6: The Five Uniform  $[1, 2, 2, 2, 1]$ -graphs

all Koszul.

□

**Theorem 4.2.** *Consider the collection of all uniform layered graphs with unique maximal and minimal elements.  $\Gamma = H$  is the minimal example producing a non-Koszul  $A(\Gamma)$ .*

*Proof.* To clarify, by minimal we mean in terms of number of vertices though it has been checked by computer that this is also the edge-minimal example. Using proposition 4.2 we only need to show that  $H$  is the unique example with nine vertices producing a non-Koszul  $A(\Gamma)$ .

To show this, we need to check the ten uniform  $[1, 3, 2, 2, 1]$ -graphs, ten uniform  $[1, 2, 2, 3, 1]$ -graphs, and twenty-three uniform  $[1, 2, 3, 2, 1]$ -graphs (see Figures 7, 8, and 9.) This was done by computer, though these 43 cases individually can still each be done by hand.

□

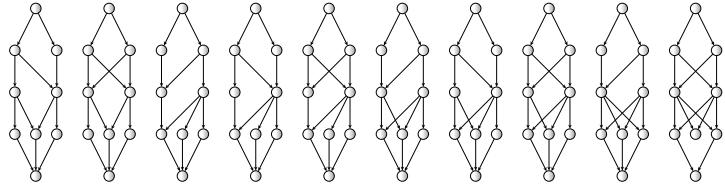


Figure 7: The Ten Uniform  $[1,3,2,2,1]$ -graphs

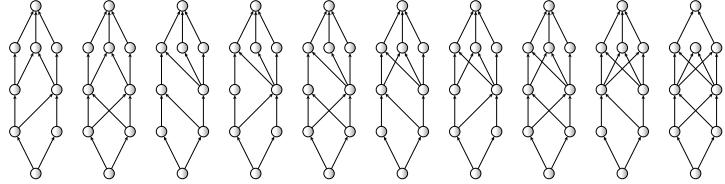


Figure 8: The Ten Uniform  $[1,2,2,3,1]$ -graphs

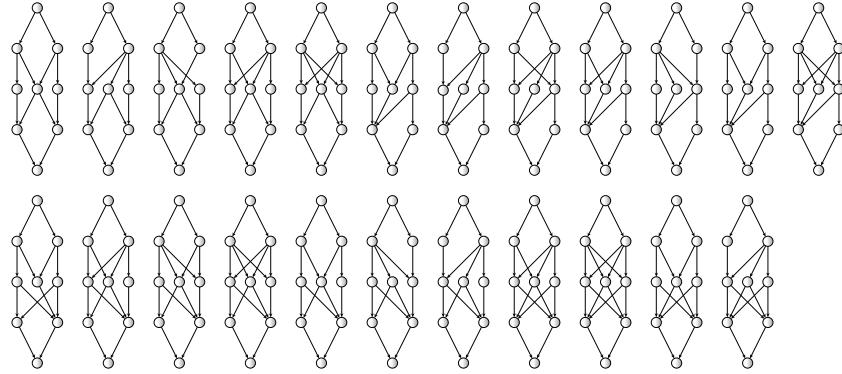


Figure 9: The Twenty-three Uniform  $[1,2,3,2,1]$ -graphs

The graph  $H$  is minimal under more general conditions as well. We can first drop the requirement that the graph must have a unique maximal element, but still require that maximal elements be of maximal rank. This changes very little, but does include the possibility of a  $[1,2,2,2,2]$ -graph with non-Koszul  $A(\Gamma)$ . There are thirty-five of these, twenty one of these being uniform, also not too time consuming for us to check.

To generalize further, we drop the last condition and allow maximal elements at every level. Now that we include graphs like the one shown in Figure 10,

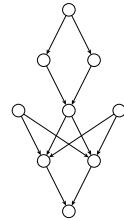


Figure 10: An Arbitrary Uniform  $[1,2,3,2,1]$ -graph

we have many more examples to consider. For ruling out  $[1,2,2,2,1]$ -graphs, there are thirty-three cases needing to be checked. To show  $H$  is minimal amongst graphs with nine vertices we must check 83  $[1,3,2,2,1]$ -graphs, 170  $[1,2,3,2,1]$ -graphs, 93  $[1,2,2,3,1]$ -graphs, and 65  $[1,2,2,2,2]$ -graphs. This is a bit much to check by hand, but it can and has been done by computer. With this, the following has been shown:

**Theorem 4.3.** Consider the collection of all uniform layered graphs with a unique minimal element.  $\Gamma = H$  is the minimal example producing a non-Koszul  $A(\Gamma)$ .

## 5 Acknowledgements

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